

ESTIMATION OF DERIVATIVES OF NONSMOOTH PERFORMANCE MEASURES IN REGENERATIVE SYSTEMS

TITO HOMEM-DE-MELLO

We investigate the problem of estimating derivatives of expected steady-state performance measures in parametric systems. Unlike most of the existing work in the area, we allow those functions to be *nonsmooth* and study the estimation of *directional derivatives*. For the class of regenerative Markovian systems we provide conditions under which we can obtain *consistent* estimators of those directional derivatives. An example illustrates that the conditions imposed must be different from those in the differentiable case. The result also allows us to derive necessary and sufficient conditions for differentiability of the expected steady-state function. We then analyze the process formed by the *subdifferentials* of the original process, and show that the subdifferential set of the expected steady-state function can be expressed as an average of integrals of multifunctions, which is the approach commonly found in the literature for integrals of sets. The latter result can also be viewed as a limit theorem for more general compact-convex multivalued processes.

1. Introduction. In recent years a great deal of attention has been devoted to the computation of *derivatives* of performance measures in stochastic systems. The information provided by those quantities is essential to answer the important question: How much will the performance change if some parameters of the system are slightly changed? More formally, suppose we have a stochastic process, say $\{X_n(\theta)\}$, depending on a (vector-valued) parameter θ , and assume that the process converges to a steady-state $X_\infty(\theta)$. We would like to compute the gradient of the expected value of the process in equilibrium, i.e., $\nabla\mathbb{E}[X_\infty(\theta)]$. A typical example is a G/G/1 queue where the distribution of the service times depends on a parameter θ (e.g., its mean); we may be interested in computing the sensitivity of the expected waiting time $\mathbb{E}[W_\infty(\theta)]$ with respect to the parameter θ . Notice also that the computation (or estimation) of derivatives allows one to take an additional step and develop *optimization* procedures for the underlying performance measure. Such effort brings obviously numerous benefits, and in fact there have been several papers in the literature dealing with that issue. See, for instance, Chong and Ramadge (1994), L'Ecuyer and Glynn (1994), Suri and Leung (1989) and references therein for description of methods and further applications.

In general, however, closed-form expressions for the steady-state derivatives cannot be obtained, so one must resort to *simulation* methods like finite differences, perturbation analysis or likelihood ratios in order to estimate gradients (see, e.g., Glasserman 1991, Glynn 1989, L'Ecuyer 1990, Rubinstein and Shapiro 1993, Suri 1989 for discussions on that topic). In addition, it is necessary to show *consistency* of such estimators, since the steady-state performance measure of the system under scrutiny is a limiting quantity and hence so is its gradient. Extra assumptions that guarantee some type of uniform convergence, such as convexity (see Hu 1992, Shapiro and Wardi 1994, Robinson 1995), are often imposed

Received September 1, 1997; revised August 28, 1998; May 24, 1999, and March 15, 2001.

MSC 2000 subject classification. Primary: 60Gxx; secondary: 90C30.

OR/MS subject classification. Primary: Probability/stochastic model; secondary: programming/nondifferentiable.

Key words. Derivative estimation, nonsmooth optimization, regenerative processes, steady-state systems, multifunctions, convex analysis.